# Incentives for network allocation with strategic agents: unifying the design process and aiming at fairness

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- PhD student: Abhinav Sinha

- Thanks to Demos Teneketzis and his former PhD student Ali Kakhbod

# Overview

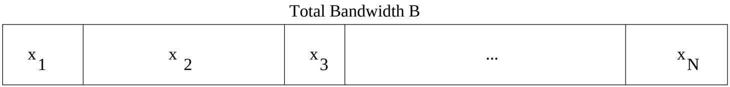
# Overview

1. Introduction, Motivation, Examples
2. Mechanism Design
3. Unified mechanism design for convex problems
4. Mechanism design for fair allocation

# Allocation problems in networks

- A number of interesting problems in networks involve optimization of a social utility function (sum of agents' utilities) under resource constraints, privacy constraints, and strategic behavior by agents
- Examples include:
  - Bandwidth allocation to cellular service providers
  - unicast, multi-rate multicast service on the Internet
  - power production/distribution/consumptionon the smart grid,
  - advertisement on social networks
  - economies with public or local public goods (e.g., investment on clean air or cyber-security)

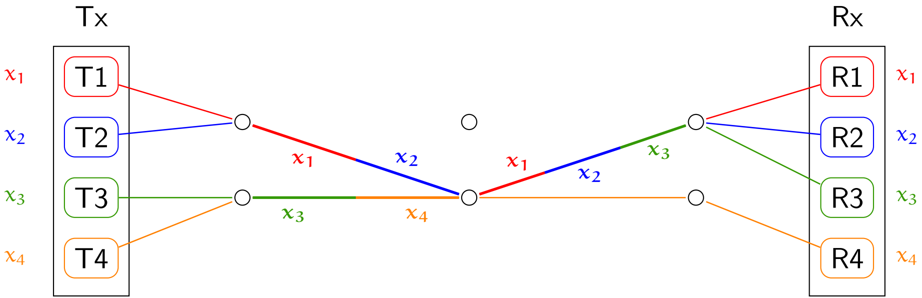
# Spectrum allocation (aka the “corn allocation” problem)




$$\max_{x \in \mathbb{R}_+^N} \sum_{i \in \mathcal{N}} v_i(x_i)$$
$$\text{s.t.} \quad \sum_{i \in \mathcal{N}} x_i \leqslant B$$

- utilities $v_i(\cdot)$ depend only on their own allocation

# Unicast service on the Internet

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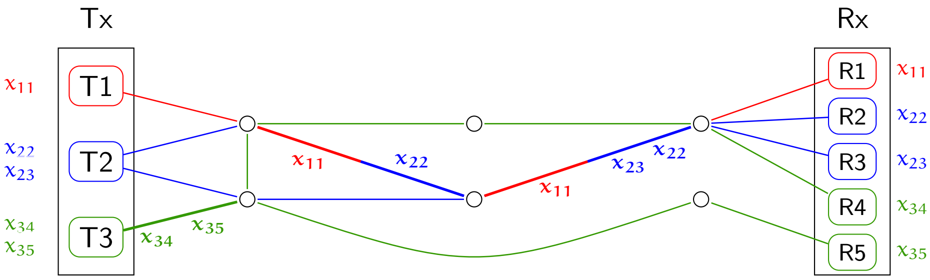
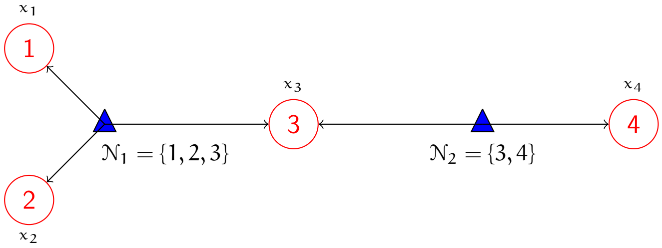



$$\max_{x \in \mathbb{R}^N_+} \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{G}_k} v_{ki}(x_{ki})$$

$$\text{s.t.} \quad \sum_{k \in \mathcal{K}^l} \max_{i \in \mathcal{G}^l_k} \{x_{ki}\} \leqslant c^l \quad \forall\, l \in \mathcal{L}$$

# Power allocation in wireless networks

- Each "agent" is a (transmitter, receiver) pair.




$$\max_{x} \sum_{i \in \mathcal{N}} v_i(\{x_i\}_{i \in \mathcal{N}_{k(i)}})$$
$$\text{s.t.} \quad x \in \mathbb{R}_+^N$$

- The vector of transmission powers $x = (x_1, \dots, x_N)$ is a public (or local public) good

# Basic approaches to solving resource allocation problems

- Two ways to solve these problems: Centralised and Decentralised.
- Centralised - Easy to solve, but...

**Two Problems**

1. Difficulty in collecting/communicating private information (e.g., utility functions $\{v_i(\cdot)\}_i$)
2. Strategic behaviour of agents - agents may lie about their private info to gain advantages!

# Simplest example: "corn allocation"

$$\max_{x} \sum_{i \in \mathcal{N}} v_i(x_i) \quad \text{s.t.} \quad \sum_{i \in \mathcal{N}} x_i \leqslant 1 \ \& \ x_i \geqslant 0 \ \ \forall\, i \in \mathcal{N}$$

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- Centralized solution (convex optimization problem)

$$L(x, \lambda, \mu) = \sum_{i \in \mathcal{N}} v_i(x_i) - \lambda \left( \sum_{i \in \mathcal{N}} x_i - 1 \right) + \sum_{i \in \mathcal{N}} \mu_i x_i$$

$$\text{(KKT)} \qquad v_i'(x_i^\star) = \lambda^\star \quad \forall\, i \in \mathcal{N} \quad \& \quad \sum_{i \in \mathcal{N}} x_i^\star = 1$$

# Decentralising the solution

- Consider the following iterative algorithm for every time $t$

Authority announces
$\lambda(t) \in \mathbb{R}_+$

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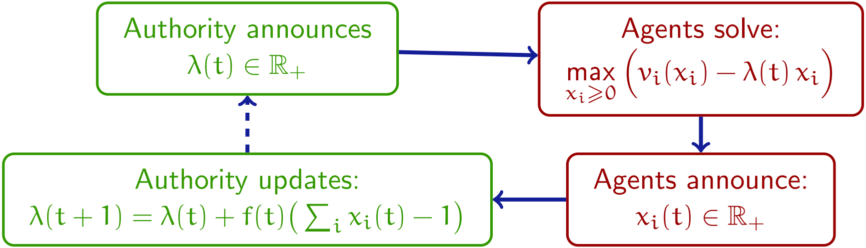

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Authority updates:
$\lambda(t+1) = \lambda(t) + f(t) \big( \sum_i x_i(t) - 1 \big)$

- Can show this converges to $\lambda(t) \to \lambda^\star$ and $x_i(t) \to x_i^\star$
- Interpret $\lambda(t)$ as **price** for corn and $\lambda(t)x_i(t)$ as **virtual tax** for $i$.

# Decentralising the solution: problems

- What if agents don't solve the pre-decided problem (e.g., announce a large demand $x_i(t)$)?
- Problem with previous (decentralized) approach - *prone to manipulation*

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- **Game Theory** is an ideal tool to model for and analyse behaviour of *strategic agents*.
- **Mechanism Design** refers to designing appropriate incentives/disincentives through taxation so that the desired centralized solution is obtained even at the presence of strategic (selfish) agents

# Overview

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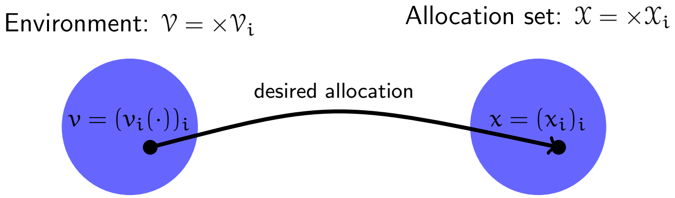

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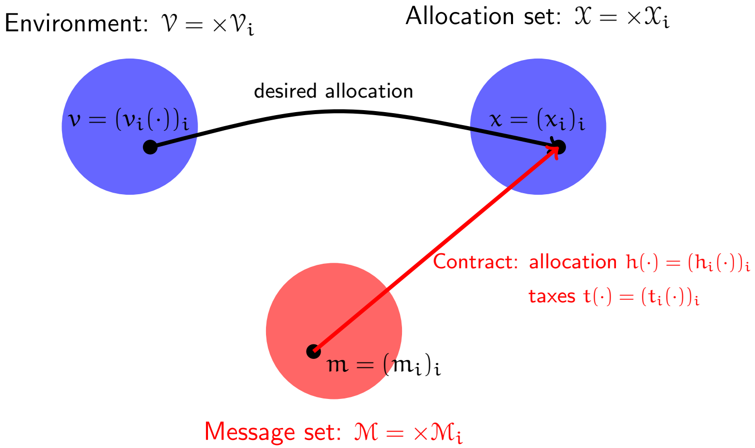

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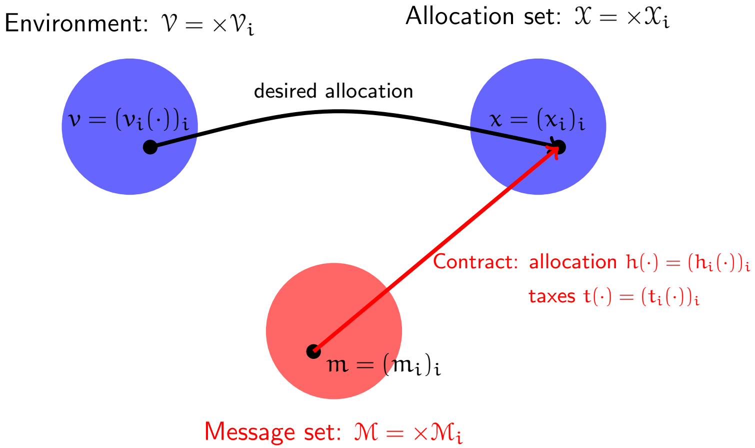


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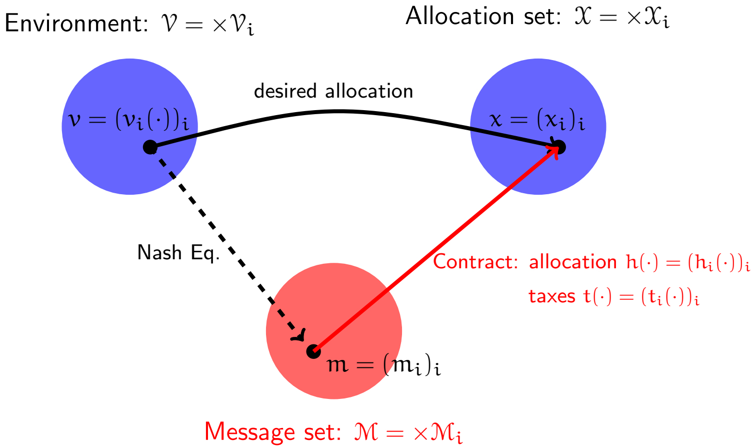


A game is induced with actions $m_i \in \mathcal{M}_i$ and utilities $u_i(m) = v_i(h_i(m)) - t_i(m)$

We desire the (Nash) equilibria $m^*$ of the induced game to map to the desired allocation

# Desired equilibrium properties

- **Design**: Message Space $\mathcal{M} = \times \mathcal{M}_i$, contract $h : \mathcal{M} \to \mathcal{X}$, taxes (or subsidies) $t : \mathcal{M} \to \mathbb{R}^N$.

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- Desired properties of any mechanism: small message space, $\mathcal{M}$
- Desired properties at equilibrium:
  - efficiency (all NE result in desired allocation)
  - individual rationality (agents weakly better off playing the game)
  - strong budget balance (SBB) at Nash equilibria (no influx of money required and no money left on the table).

# Desired off-equilibrium properties

Why do we need them in the first place?

- Any practical implementation requires some kind of "learning" process
- In the course of such process, agents: quote messages, receive allocations, are taxed/subsidised, learn each others' utilities, until they converge to Nash Eq.

# Desired off-equilibrium properties

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Desired properties (in decreasing order of significance)

1. Holly grail: Stability (NE is the guaranteed limit for a *large* class of "learning" processes)
2. Feasibility off equilibrium (otherwise we cannot allocate during "learning" process–contract promises but cannot deliver!)
3. Also useful to have SBB off equilibrium: at every stage no leftover money to redistribute inter-temporally

# Overview

# State of the art

- Main criticism with state of the art solutions
  1. Proposed mechanisms are fragmented, case-by-case (there is no unified design) thus obscuring fundamental understanding of these problems
  2. Almost all of them are not stable[1]
  3. Surprisingly most of them are not even "learning"-ready:
     – either use 1D messages (proven to be insufficient for stability),
     – or are not feasible off equilibrium!

[1] Notable exception [Healy and Mathevet, 2012] that introduce globally stable "contractive" mechanisms (not supermodular)

# One problem to rule them all: “corn allocation”

Total Bandwidth B

| | | | | |
|---|---|---|---|---|
| $x_1$ | $x_2$ | $x_3$ | ... | $x_N$ |

$$
\begin{aligned}
\max_{x \in \mathbb{R}_+^N} \quad & \sum_{i \in \mathcal{N}} v_i(x_i) \\
\text{s.t.} \quad & \sum_{i \in \mathcal{N}} x_i \leqslant B
\end{aligned}
$$

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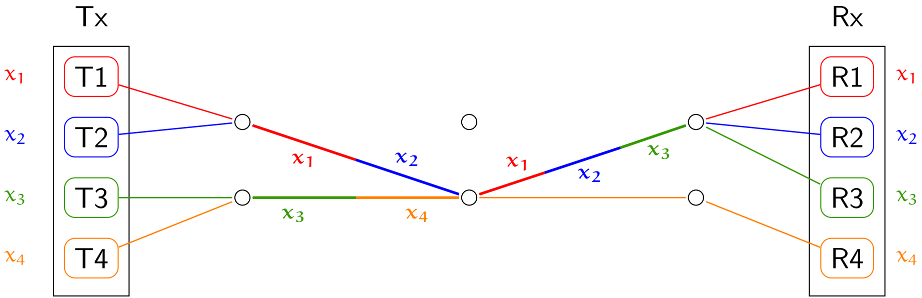


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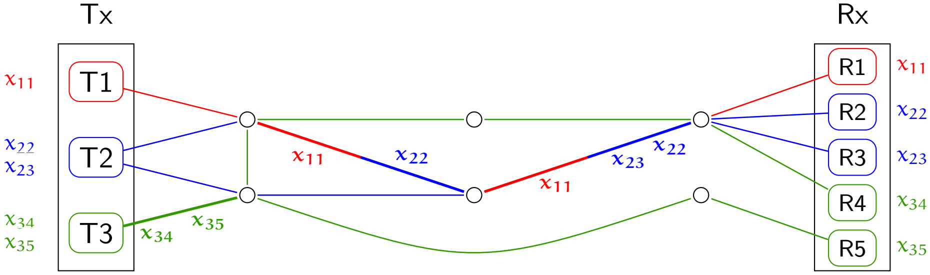


$$\max_{x \in \mathbb{R}_+^N} \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{G}_k} v_{ki}(x_{ki})$$

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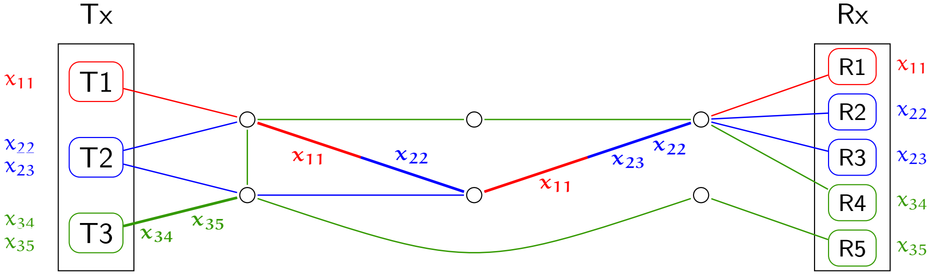


Introduce auxiliary variables $s_k^l \stackrel{\text{def}}{=} \max_{i \in \mathcal{G}_k^l} \{x_{ki}\}$

$$\max_{(x,s)\in\mathbb{R}_+^{N+S}} \quad \sum_{ki\in\mathcal{N}} v_{ki}(x_{ki})$$

$$\text{s.t.} \quad \sum_{k\in\mathcal{K}^l} s_k^l \leqslant c^l \quad \forall\, l \in \mathcal{L},$$

$$\text{s.t.} \quad x_{ki} \leqslant s_k^l \quad \forall\, i \in \mathcal{G}_k^l,\ k \in \mathcal{K}^l,\ l \in \mathcal{L}.$$

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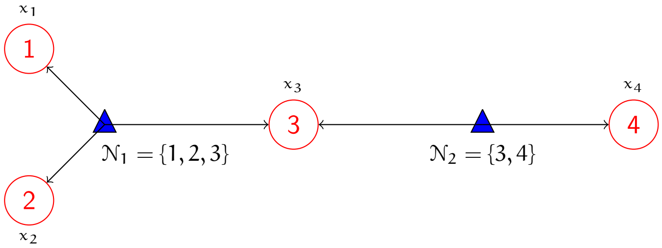


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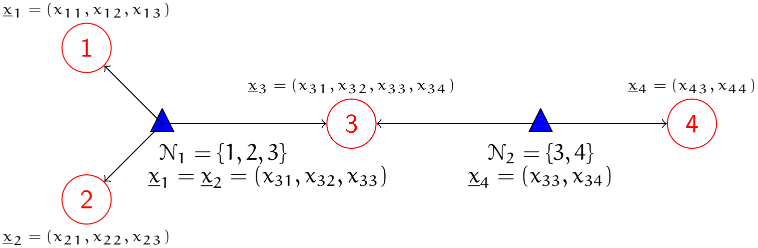


Vectorize variables to turn public goods into private goods with equality constraints

$$\max_{x} \sum_{i \in \mathcal{N}} v_i(\underline{x}_i)$$

$$\text{s.t.} \quad \underline{x}_i \in \mathbb{R}_+^{D_i} \quad \forall\, i \in \mathcal{N},$$

$$\text{s.t.} \quad E_i^k \underline{x}_i = E_j^k \underline{x}_j \quad \forall\, i, j \in \mathcal{N}_k,\ \forall\, k \in \mathcal{K}.$$

# General Centralized Problem

What is common in all these problems?

[2] For ease of exposition we only consider scalar $x_i$'s and incorporate equality with inequality constraints

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What is common in all these problems?
Optimization of sum of **separable** concave functions under linear inequality/equality constraints[2]

$$\max_{x \in \mathbb{R}_+^N} \sum_{i \in \mathcal{N}} v_i(x_i) \qquad \text{(CP)}$$
$$\text{s.t.} \quad A_l^\top x \leqslant c_l \ \ \forall\, l \in \mathcal{L}$$

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# Proposed Mechanism

- Each user quotes the message $m_i = (y_i, p_i)$, where:
  $y_i$ is the desired demand,
  $p_i = \left(p_i^l\right)_{l \in \mathcal{L}_i}$ with $p_i^l \in \mathbb{R}_+$ being the suggested price for $l$-th constraint.

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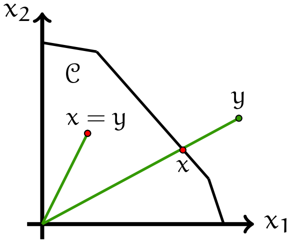
- Allocation is generated so that $y$ maps continuously to a point $x$ in the feasibility set $\mathcal{C}$, i.e., $x = h(y, p) = h(y) \in \mathcal{C}$.



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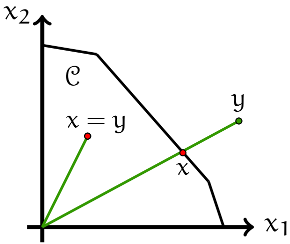


- With equality constraints, same idea but in the non-degenerate projection of $\mathcal{C}$

# Proposed Mechanism (cont.)

- Taxes: $t_i(y,p) = \sum_{l \in \mathcal{L}_i} t_i^l(y,p)$, with

$$t_i^l(y,p) = \underbrace{A_{li}\bar{p}^l_{-i}x_i}_{\text{stationarity}} + \underbrace{\left(p_i^l - \bar{p}^l_{-i}\right)^2}_{\text{equal prices}} + \underbrace{\eta\,\bar{p}^l_{-i}p_i^l\left(c_l - A_l^\top x\right)^2}_{\text{complementary slackness}} + \underbrace{\Delta_i^l}_{\text{budget balance}}$$

  and $\bar{p}^l_{-i}$ is the average price quoted for constraint $l$ by all other users.
- Purpose of each tax term is to satisfy each KKT condition (of the centralized problem) at equilibrium

# Basic Result

**Proposition**

*Assuming strictly concave and differentiable utilities + technical assumptions for corner cases, the proposed mechanism is*

- *efficient, individually rational, and strong budget balanced at Nash equilibria*

*In addition, the mechanism is*

- *feasible, and strong budget balanced off equilibrium*

# Summary, current and future work

- We have a general mechanism template for convex allocation problems
- Proposed mechanisms are "learning"-ready but **not stable** yet: we are working on it.

# Overview

# Before we start: a notational change

- So far agents' utilities were arbitrary functions $v_i(\cdot)$ (which were their private information),
- We now refer to them as $v(\cdot, \theta_i)$, where $v$ is a known function and $\theta_i \in \Theta_i$ is the **type** of agent $i$ (which is his private info)
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- All mechanisms will now be **direct** mechanisms, i.e., each user's message is $\phi_i \in \Theta_i$.
- So each user quotes (possibly untruthfully) his type.
- Mechanism design boils down to incentivising users to report their type truthfully

# Social welfare measured as sum of utilities (SoU)

- So far we tacitly assumed that the social utility is sum of utilities (SoU) of individual agents.

$$V(x;\theta) = \sum_{i=1}^{N} v(x;\theta_i)$$

- Such a choice is intuitive and justified (e.g., is a symmetric function)...
- ...but clearly not the only one of interest to a social planner, especially if fairness of allocation is important $(100 + 0 = 50 + 50)$.

# Social welfare: a brief history

- "The community is a fictitious body, composed of the individual persons who are considered as constituting as it were its members. The interest of the community then is, what?–the sum of the interests of the several members who compose it."

*"An Introduction to the Principles of Morals and Legislation", Jeremy Bentham, 1823*

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- "Social and economic inequalities are to be arranged so that they are both [...] to the greatest benefit of the least advantaged"

*"A Theory of Justice", John Rawls, 1971*

# Social utility for fair allocation

- Different forms of social utility have been proposed in place of SoU:

$$\min_{i \in \mathcal{N}} v(x; \theta_i)$$

$$\sum_{i \in \mathcal{N}} \big(v(x; \theta_i)\big)^p \qquad \text{(for } p < 1\text{)}$$

$$\sum_{i \in \mathcal{N}} g\big(v(x; \theta_i)\big) \qquad \text{(for some concave } g(\cdot)\text{)}$$

- Why is it that mechanism designs for such forms have hardly been studied?

# SoU: a mathematically convenient formulation

Political/social/ethical reasons aside, there are strong methodological reasons why SoU is a preferred choice for mechanism design.

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Political/social/ethical reasons aside, there are strong methodological reasons why SoU is a preferred choice for mechanism design.

- Amenable to aligning objective function with strategic optimization. E.g., in direct mechanisms with allocation $\hat{x}(\phi)$,
  social utility:

$$\sum_j v(\widehat{x}(\phi); \theta_j) = v(\widehat{x}(\phi); \theta_i) + \sum_{j \neq i} v(\widehat{x}(\phi); \theta_j)$$

  vs. individual utility for agent $i$:

$$u(\phi; \theta_i) = v(\widehat{x}(\phi); \theta_i) - t_i(\phi)$$

- E.g., VCG sets $t_i(\phi) = -\sum_{j \neq i} v(\widehat{x}(\phi); \phi_j)$ to “align” incentives.

# SoU: a mathematically convenient formulation

However if we adopt $\sum_{i \in \mathcal{N}} g(v(x_i;\theta_i))$ as the social utility we have

- social utility:

$$\sum_j g(v(\widehat{x}(\phi);\theta_j)) = g(v(\widehat{x}(\phi);\theta_i)) + \sum_{j \neq i} g(v(\widehat{x}(\phi);\theta_j))$$

- vs. individual utility for agent $i$:

$$u(\phi;\theta_i) = v(\widehat{x}(\phi);\theta_i) - t_i(\phi)$$

- and after setting $t_i(\phi) = -\sum_{j \neq i} g(v(\widehat{x}(\phi);\phi_j))$

$$u(\phi;\theta_i) = v(\widehat{x}(\phi);\theta_i) + \sum_{j \neq i} g(v(\widehat{x}(\phi);\phi_j))$$

- which does not “align” individual incentives with social utility!

# The Centralized Problem with discrete types

- Consider the "corn allocation" problem (easily generalized)
- Each agent's utility $v(x_i;\theta_i)$ is described by a *private type* $\theta_i \in \Theta_i$, with $\Theta_i$ finite
- Each agent has prior $p_i \in \Delta(\Theta)$, $\Theta = \times_{i=1}^{N}\Theta_i$ on the type profile.

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- Each agent has prior $p_i \in \Delta(\Theta)$, $\Theta = \times_{i=1}^N \Theta_i$ on the type profile.
- Centralized problem

$$\widehat{x}(\theta) \triangleq \arg\max_{x \in \mathcal{X}} \sum_{i \in \mathcal{N}} g_\epsilon(v(x_i;\theta_i))$$

$$\mathcal{X} = \left\{ x \in \mathbb{R}_+^N \mid \sum_{i \in \mathcal{N}} x_i \leqslant 1 \right\}$$

- In particular we consider $g_\epsilon(v) = v - \epsilon v^2$, where the parameter $\epsilon \geqslant 0$ controls departure from SoU.
- Note: $\epsilon$ is bounded above by the requirement that $g_\epsilon(v(x_i;\theta_i))$ is monotonic and concave function of $x_i \in [0,1]$

# Mechanism Design Problem (Bayesian)

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- Allocation: For message profile $\phi = (\phi_i)_{i \in \mathcal{N}}$, allocation is $\widehat{x}(\phi)$

# Mechanism Design Problem (Bayesian)

- Messages: Direct Mechanism i.e. $m_i = \phi_i \in \Theta_i \ \forall\ i$.
- Allocation: For message profile $\phi = (\phi_i)_{i \in \mathcal{N}}$, allocation is $\widehat{x}(\phi)$
- Taxes: It is desirable to design $\{t_i(\phi)\}_{i \in \mathcal{N}, \phi \in \Theta}$ such that *Incentive compatibility* (IC) constraints are satisfied.
  $\forall\ i \in \mathcal{N},\ \forall\ \theta_i, \psi_i \in \Theta_i$:

$$\sum_{\theta_{-i}} p_i(\theta_{-i}|\theta_i)\left[v(\widehat{x}_i(\theta_i, \theta_{-i}); \theta_i) - t_i(\theta_i, \theta_{-i})\right]$$
$$\geqslant \sum_{\theta_{-i}} p_i(\theta_{-i}|\theta_i)\left[v(\widehat{x}_i(\psi_i, \theta_{-i}); \theta_i) - t_i(\psi_i, \theta_{-i})\right]$$

- For Bayesian setup, this would guarantee truth telling as a Bayesian-NE.

# Main Result

**Proposition**

- *For any set of priors* $\{p_i\}_{i \in \mathcal{N}}$ *and for any* $\epsilon \in [0, \epsilon_{max}]$ *there exist taxes* $\{t_i(\phi)\}_{i \in \mathcal{N}, \phi \in \Theta}$ *that satisfy the IC conditions.*
- *For the case of two-type agents (i.e.,* $|\Theta_i| = 2, \forall i \in \mathcal{N}$*), implementation in dominant strategies is possible.*

Note: The value of $\epsilon_{max}$ is universal wrt priors $\{p_i\}_{i \in \mathcal{N}}$ (depends only on utility functions)

# Outline of Proof

- IC conditions involve $N|\Theta_i|^{N+1}$ inequalities and $N|\Theta_i|^N$ variables.
- Decomposition possible: for a given $i, \phi_{-i}$, there are $|\Theta_i|^2$ inequalities and $|\Theta_i|$ variables $\{t_i(\phi_i, \phi_{-i})\}_{\phi_i \in \Theta_i}$.

[3] Original ideas from seminal work by D'Aspremont et.al. 1979,1990,2003.

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- Farkas Lemma[3]: The system

$$x \in \mathbb{R}^N \quad \text{with} \quad Ax \leqslant b \quad \text{for} \quad A \in \mathbb{R}^{M \times N},\ b \in \mathbb{R}^M$$

is feasible in $x$ iff all $\lambda \in \mathbb{R}_+^M$ that satisfy $A^\top \lambda = 0$ also satisfy $b^\top \lambda \geqslant 0$.

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- For $\epsilon = 0$ the conditions are satisfied[1]
- For $\epsilon > 0$, proof relies upon the fact that discrete types lead to "sufficiently distant" optimal solutions.

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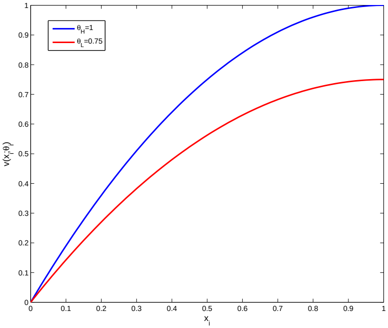
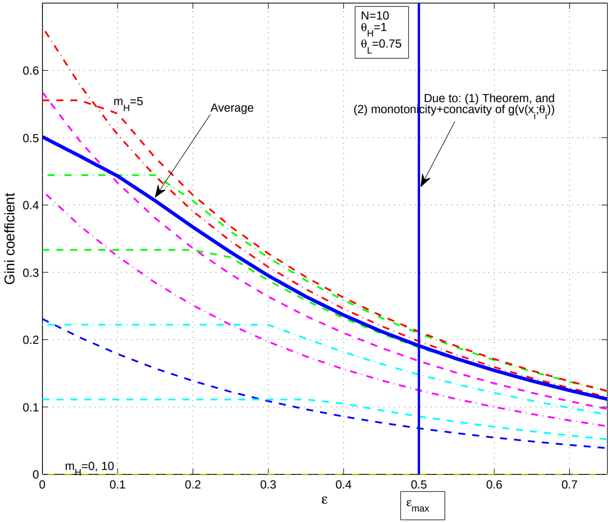
# Numerical Example

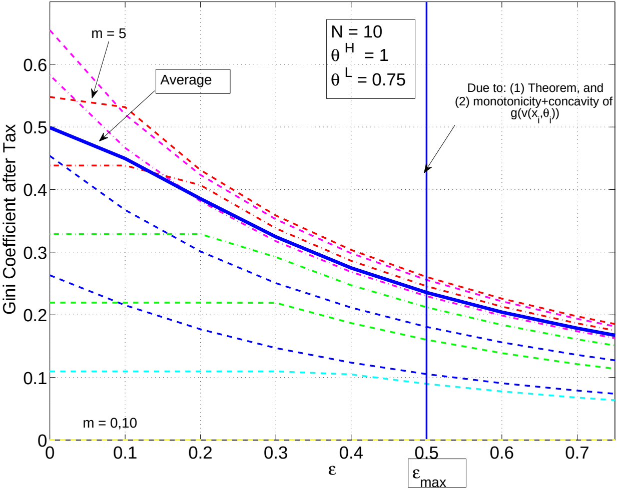
- Q: Is $\epsilon_{max}$ (from Theorem) sufficiently large to induce a substantially more fair reallocation of resources?
- A: Depends on the specifics of the problem
- Example: Two types (high, low) $\Theta_i = \{\theta^H, \theta^L\}$ and utility $v(x_i;\theta_i) = 2\theta_i x_i - \theta_i x_i^2$



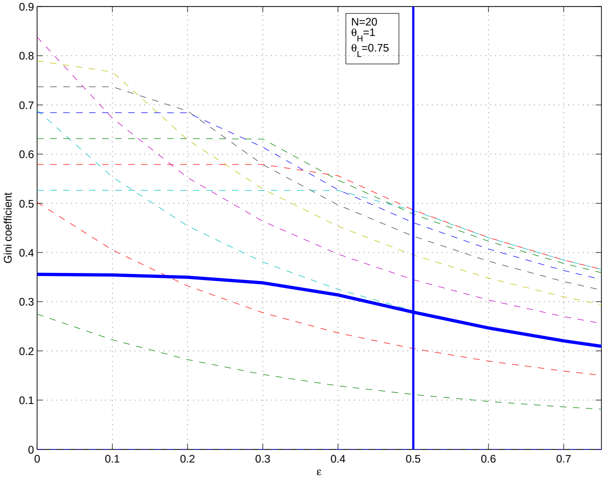
# Numerical Example (N=10)



# Numerical Example (N=10) – After Tax



# Numerical Example (N=20)



# Current/Future Work

1. General methodology for more aggressive $g_\epsilon(\cdot)$ functions (e.g., $g_\epsilon(v) = v^{1-\epsilon}$) or even arbitrary functions $g_\epsilon(\cdot)$
2. Full implementation (have some partial results: can be done by adding one more message per user)
3. Budget balanced versions (based on AGV tax structures–similar techniques apply)

Thank you!

Healy, P. J. and Mathevet, L. (2012).
Designing stable mechanisms for economic environments.
*Theoretical Economics*, 7(3):609–661.