Problems in the intersection of Information theory and Control

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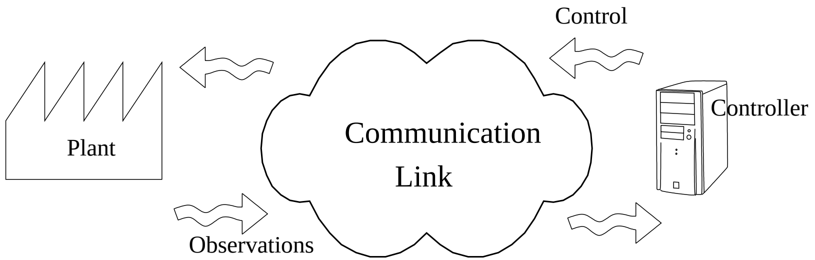
- Former PhD student: Junghuyn Bae
- Current PhD students: Deepanshu Vasal, Jui Wu

- Thanks to Demos Teneketzis and his former students Aditya Mahajan and Ashutosh Nayyar

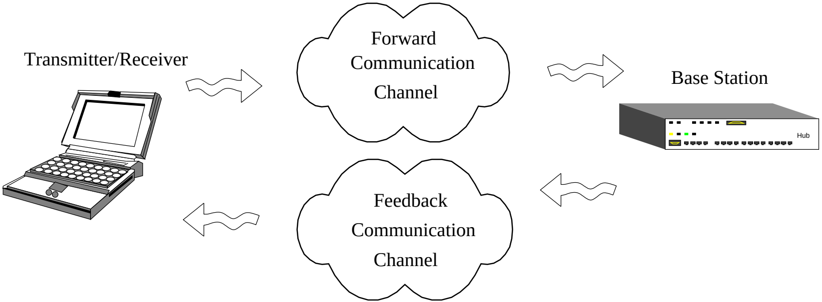
What this talk is NOT about

Controlling of a plant from a distance through a communication link



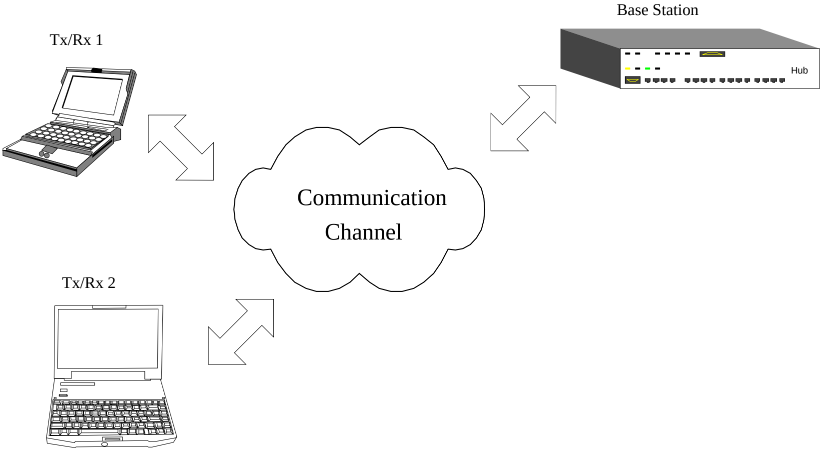
Instead, we are interested in

A) Viewing point-to-point communication as a Control problem



The act of transmitting a signal (partially) controls the overall communication system, with the hope of bringing it to a "desirable" state

B) Viewing multi-agent communications as a Control problem



Multiple agents (partially) control a communication network to bring it to a state beneficial for all (cooperatively/competitively)

C) More subtle: Viewing off-line optimization problems relevant to Information theory as control problems, e.g., Shannon capacity

$$C = \sup_{\{P_{X_t|X^{t-1},Y^{t-1}}(\cdot|\cdot,\cdot)\}_t} \frac{1}{T}\sum_{t=1}^{T} I(X_t \wedge Y_t|Y^{t-1})$$

Only connection to Communications is the problem origin.

No clear connection to Control either.
Where is the controller? where is the plant? what is the observation/control action?

Connections between Communication and Control

- Whenever there is feedback there is an intimate relation between Communication and Control
- One possible classification of problems
  1. Use Control techniques to design transmission schemes that achieve general performance measures (e.g., real-time coding[1] $\rightarrow$ examples A,B)
  2. Use Control techniques to design transmission schemes that achieve certain information-theoretic-inspired measures (e.g., capacity, error exponents[2] $\rightarrow$ examples A,B)
  3. Use Control techniques to evaluate information-theoretic quantities (e.g., capacity, error exponents[3] $\rightarrow$ example C)

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[1] [Walrand and Varaiya, 1983, Mahajan and Teneketzis, 2009]

[2] [Horstein, 1963, Schalkwijk and Kailath, 1966, Shayevitz and Feder, 2008, Bae and Anastasopoulos, 2012]

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Connections between Communication and Control

- Another possible classification of problems
  1. Problems involving a single controller (e.g., point-to-point transmission)
  2. Problems involving multiple controllers (e.g., multi-user transmission)
     - a) Agents act as members of a **team** trying to achieve a common goal
     - b) Agents act strategically having individual goals (**games**)

- Generally, dynamic problems with multiple agents and/or strategic interaction are more difficult: no standard solution methodology

- In this presentation we will discuss centralized/decentralized sequential team problems, and a sequential problem with strategic interaction (game)

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Overview

1. Point-to-point channels with memory and noiseless feedback
2. Multiple access channel with noiseless feedback
3. Cooperative communications in relay networks

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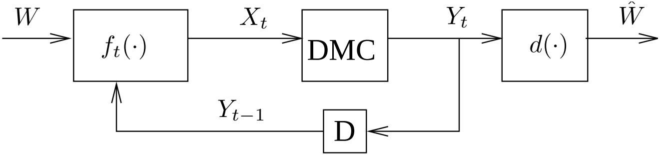
Background: DMC with noiseless feedback



- Information message $W \in \{1, 2, \ldots, 2^{nR}\}$
- Transmitted symbols $X_t \in \mathscr{X}$, $t = 1, 2, \ldots, n$
- Received symbols $Y_t \in \mathscr{Y}$, $t = 1, 2, \ldots, n$
- Discrete-memoryless channel (DMC) defined by $Q(y_t|x_t)$
- Message estimate $\hat{W} \in \mathscr{W}$
- Encoding functions $X_t = f_t(W, Y^{t-1})$, $t = 1, 2, \ldots, n$
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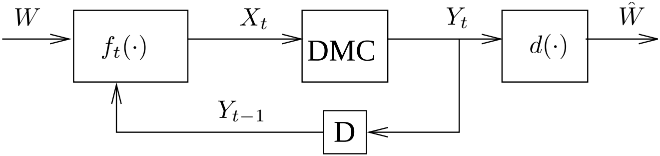
Background: DMC with noiseless feedback



Fact[Shannon]: feedback does not increase capacity!

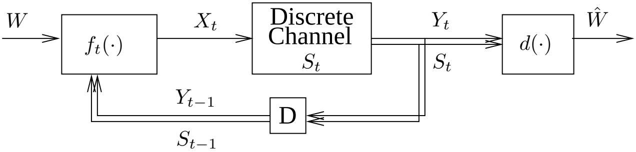
Capacity given by an **off-line, static, single-letter** optimization problem over distributions on $\mathscr{X}$

$$C = \sup_{P_X(\cdot)} \quad I(X_t \wedge Y_t),$$

with mutual information evaluated as

$$I(X_t \wedge Y_t) \stackrel{\text{def}}{=} \sum_x \sum_y P_X(x) Q(y|x) \log \frac{Q(y|x)}{\sum_{x'} P_X(x') Q(y|x')}$$

Channel with memory and noiseless feedback



- Information message $W \in \{1,2,\ldots,2^{nR}\}$
- Transmitted symbols $X_t \in \mathscr{X}$, $t = 1,2,\ldots,n$
- Channel state $S_t \in \mathscr{S}$, $t = 1,2,\ldots,n$
- Received symbols $Y_t \in \mathscr{Y}$, $t = 1,2,\ldots,n$ and channel state (perfect Rx CSI)
- Finite state channel (FSC) defined by $Q_y(y_t|x_t,s_t)$, $Q_s(s_{t+1}|s_t,x_t)$
- Message estimate $\hat{W} \in \mathscr{W}$
- Encoding functions $X_t = f_t(W,Y^{t-1},S^{t-1})$, $t = 1,2,\ldots,n$ (delayed Tx CSI)
- Decoding function $\hat{W} = d(Y^n,S^n)$

FSC capacity

Capacity of this channel is the result of the following **off-line** optimization problem[4] over infinitely many conditional distributions on $\mathscr{X}$

$$C = \sup_{\{P_{X_t|X_{t-1},S^{t-1},Y^{t-1}}\}_{t=1}^{\infty}} \liminf_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} I(X_t, X_{t-1} \wedge S_t, Y_t | S^{t-1}, Y^{t-1}).$$

- Observe: $P_{X_t|X_{t-1},S^{t-1},Y^{t-1}} \in \mathscr{X} \times \mathscr{S}^{t-1} \times \mathscr{Y}^{t-1} \to \mathscr{P}(\mathscr{X})$, so its domain increases with t
- **Q**: How can we utilize Control theory to solve this problem?

[4] [Tatikonda and Mitter, 2009, Bae and Anastasopoulos, 2010]

Parenthesis: Markov Decision Processes in 5 mins

A Markov Decision Process (MDP) is a random process with:

State $S_t \in \mathscr{S}$,

Control action $U_t \in \mathscr{U}$,

Instantaneous reward $R_t \in \mathscr{R}$,

defined by the following dynamics

$$P(S_{t+1}|S^t, U^t, R^t) = Q_s(S_{t+1}|S_t, U_t)$$
$$P(R_t|S^t, U^t, R^{t-1}) = Q_r(R_t|S_t, U_t)$$
$$U_t = g_t(S^t)$$

Problem: Design the sequence of mappings $g = \{g_t\}_t$ to maximize the average reward

$$J(g) \stackrel{\text{def}}{=} \mathbf{E}^g\{\sum_t R_t\}$$

Parenthesis: Markov Decision Processes in 5 mins

- A **single** controler observes **perfectly** the state and takes an action (centralized control with perfect state observation)

- **Solution**: Optimal control policy is Markov, i.e.,

$$U_t = g_t^*(S^t) = g_t^*(S_t)$$

- **Interpretation**: If state is perfectly observed by single controller, then it perfectly summarizes the entire history of observations

Parenthesis: MDPs and POMDPs in 5 mins

A Partially Observed MDP (POMDP) is a random process with:

State	$S_t \in \mathscr{S}$,
Observation	$Y_t \in \mathscr{Y}$,
Control action	$U_t \in \mathscr{U}$,
Instantaneous reward	$R_t \in \mathscr{R}$,

defined by the following dynamics

$$P(S_{t+1}|S^t, U^t, R^t, Y^t) = Q_s(S_{t+1}|S_t, U_t)$$
$$P(Y_t|S^t, U^{t-1}, R^{t-1}, Y^{t-1}) = Q_y(Y_t|S_t)$$
$$P(R_t|S^t, U^t, R^{t-1}, Y^t) = Q_r(R_t|S_t, U_t)$$
$$U_t = g_t(Y^t)$$

Problem: Design the sequence of mappings $g = \{g_t\}_t$ to maximize the average reward

$$J^g \stackrel{\text{def}}{=} \mathbf{E}^g\{\sum_t R_t\}$$

Parenthesis: MDPs and POMDPs in 5 mins

- **Solution**: Optimal control policy has the structure,

$$U_t = g_t^*(Y^t) = g_t^*(\Pi_t)$$

where $\Pi_t \in \mathscr{P}(\mathscr{S})$ and $\Pi_t(s) \stackrel{\text{def}}{=} Pr(S_t = s | U^{t-1}, Y^t) \quad \forall s \in \mathscr{S}$

- **Interpretation**: If state is imperfectly observed by controller, then the posterior belief of the state, Π_t, perfectly summarizes the entire history of observations
- **Takeaway:** MDPs/POMDPs are useful tools when we want to summarize the dependence of previous observations in our present decisions

Back to our problem: FSC capacity

$$C = \sup_{\{P_{X_t|X_{t-1},S^{t-1},Y^{t-1}}\}_{t=1}^{\infty}} \liminf_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} I(X_t, X_{t-1} \wedge S_t, Y_t | S^{t-1}, Y^{t-1}).$$

- Think of C as the total average reward of some "fictitious" MDP with appropriate states, actions, instantaneous rewards, etc
- Hint:

$$I(X_t, X_{t-1} \wedge S_t, Y_t | S^{t-1}, Y^{t-1}) =$$

$$= \mathbf{E}\{\log \frac{P(S_t, Y_t | X_t, X_{t-1}, S^{t-1}, Y^{t-1})}{P(S_t, Y_t | S^{t-1}, Y^{t-1})}\}$$

$$= \mathbf{E}\{\log \frac{Q_y(Y_t|S_t, X_t) Q_s(S_t|S_{t-1}, X_{t-1})}{\sum_{x_t, x_{t-1}} Q_y(Y_t|S_t, x_t) Q_s(S_t|S_{t-1}, x_{t-1}) P(x_t|x_{t-1}, S^{t-1}, Y^{t-1}) P(x_{t-1}|S^{t-1}, Y^{t-1})}\}$$

Back to our problem: FSC capacity

Instantaneous reward depends on

a) some current variables, e.g., $S_{t-1}, S_t, X_{t-1}, X_t, Y_t$

b) the input distribution $P(x_t|x_{t-1}, S^{t-1}, Y^{t-1})$

c) the quantity $P(x_{t-1}|S^{t-1}, Y^{t-1})$

- Define the Control action $U_t \in \mathscr{X} \to \mathscr{P}(\mathscr{X})$ (conditional distribution of X_t given X_{t-1})
- Allow U_t to be a deterministic function of S^{t-1}, Y^{t-1}, $U_t = g_t(S^{t-1}, Y^{t-1})$
- Meaning: $U_t(x_{t-1})(\cdot) = g_t(S^{t-1}, Y^{t-1})(x_{t-1})(\cdot) = P(X_t = \cdot|X_{t-1} = x_{t-1}, S^{t-1}, Y^{t-1})$

FSC capacity

- Further define the r.v. (information state) $\Theta_t \in \mathscr{P}(\mathscr{X})$ with $\Theta_t(x) \stackrel{\text{def}}{=} Pr(X_t = x | S^t, Y^t), \quad \forall x \in \mathscr{X}$
- Average instantaneous reward becomes

$$I(X_t, X_{t-1} \wedge S_t, Y_t | S^{t-1}, Y^{t-1}) =$$

$$= \mathbf{E}\{\log \frac{Q_y(Y_t|S_t, X_t) Q_s(S_t|S_{t-1}, X_{t-1})}{\sum_{x_t, x_{t-1}} Q_y(Y_t|S_t, x_t) Q_s(S_t|S_{t-1}, x_{t-1}) U_t(x_{t-1})(x_t) \Theta_{t-1}(x_{t-1})}\}$$

- Observe: dependence on S^{t-1}, Y^{t-1} is "hidden" in the **generation** of $U_t = g_t(S^{t-1}, Y^{t-1})$ and the **evolution** of Θ_{t-1}

FSC capacity

Theorem ([Bae and Anastasopoulos, 2010])

The original optimization problem is equivalent to an MDP with

- *State* $\{(S_{t-1}, \Theta_{t-1})\}_t$
- *Control action* U_t

Markov policies are optimal, i.e., optimal actions can be of the form

$$U_t = g_t^*(S_{t-1}, \Theta_{t-1}) \Leftrightarrow P^*_{X_t|X_{t-1}, S^{t-1}, Y^{t-1}} = P^*_{X_t|X_{t-1}, S_{t-1}, \Theta_{t-1}}$$

Capacity is now simplified to a **single-letter** *expression*

$$C = \sup_{P_{X_t|X_{t-1}, S_{t-1}, \Theta_{t-1}}} I(X_t, X_{t-1} \wedge S_t, Y_t | S_{t-1}, \Theta_{t-1})$$

Lessons learned

- Complex optimization problems in Information theory can be translated to simple centralized control problems
- Starting point: some multi-letter capacity expression
- Methodology: appropriately define a control system to unveil an MDP/POMDP
- These ideas can also help in designing actual on-line capacity-achieving transmission schemes (not in this talk)

Overview

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2. Multiple access channel with noiseless feedback
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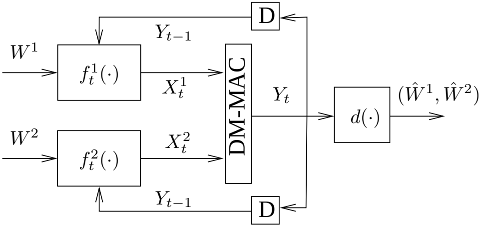
System model: the information theoretic setup



- Messages $W^i \in \{1, 2, \ldots, 2^{nR^i}\}$, $i = 1, 2$
- Transmitted symbols $X_t^i \in \mathscr{X}^i$, $i = 1, 2$, $t = 1, 2, \ldots, n$
- Received symbols $Y_t \in \mathscr{Y}$, $t = 1, 2, \ldots, n$
- Discrete-memoryless MAC (DM-MAC) $Q(y_t | x_t^1, x_t^2)$
- Message estimates $(\hat{W}^1, \hat{W}^2) \in \mathscr{W}^1 \times \mathscr{W}^2$
- Encoding functions $X_t^i = f_t^i(W^i, Y^{t-1})$, $i = 1, 2$, $t = 1, 2, \ldots, n$
- Decoding function $(\hat{W}^1, \hat{W}^2) = d(Y^n)$

NLF-MAC capacity

- Feedback provides an enlargement of capacity region for the MAC [Gaarder and Wolf, 1975, Cover and Leung, 1981, Ozarow, 1984, Bross and Lapidoth, 2005, Venkataramanan and Pradhan, 2009]
- For the general NLF-MAC the capacity is not known as a single-letter expression!
- Multi-letter expression was developed by [Kramer, 2003]

NLF-MAC capacity: multi-letter expression

Fact ([Kramer, 2003], [Salehi, 1978])

The problem of evaluating the NLF-MAC capacity region is equivalent to solving the following optimization problem for every $\underline{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \geq 0$

$$J_{\underline{\lambda}} = \sup \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \{\lambda_1 I(X_t^1 \wedge Y_t | X^{2,t}, Y^{t-1}) + \lambda_2 I(X_t^2 \wedge Y_t | X^{1,t}, Y^{t-1}) + \lambda_3 I(X_t^1, X_t^2 \wedge Y_t | Y^{t-1})\},$$

and the supremum is over all input distributions of the form

$$\{P(X_t^1 | X^{1,t-1}, Y^{t-1}),\ P(X_t^2 | X^{2,t-1}, Y^{t-1})\}_t$$

Why is the NLF-MAC capacity still an open problem?

- Three main difficulties
  1. The optimal input distributions on X_t^i depend on entire history $X^{i,t-1}$ and Y^{t-1}
  2. The optimization problem involves two controllers with different observations (decentralized control!)
  3. The per-stage rewards (mutual info expressions) are complicated functions of the involved random variables
- Claim: we can address the first two of the above three difficulties
- Solve a slightly different problem: real-time communication over the NLF-MAC

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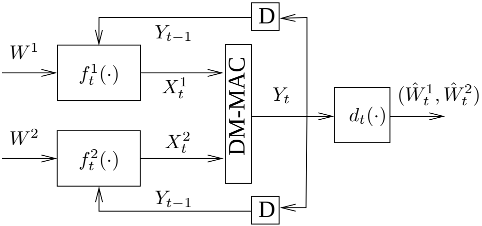
System model: real-time communication



Same model as before, except

- Message estimates for each time t, $(\hat{W}_t^1, \hat{W}_t^2) \in \mathscr{W}^1 \times \mathscr{W}^2$
- Decoding functions $(\hat{W}_t^1, \hat{W}_t^2) = d_t(Y^t)$, $t = 1, 2, \ldots, n$
- Instantaneous reward function $\rho_t(W^1, W^2, \hat{W}_t^1, \hat{W}_t^2)$
- Find a set of encoding/decoding functions $g \stackrel{\text{def}}{=} \{f_t^1, f_t^2, d_t\}_t$ that maximize

$$J(g) = \mathbf{E}^g\{\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \rho_t(W^1, W^2, \hat{W}_t^1, \hat{W}_t^2)\},$$

Problem statement: discussion

- Many reasonable choices for reward functions $\rho_t(\cdot)$, e.g.,

$$\rho_t(W^1, W^2, \hat{W}_t^1, \hat{W}_t^2) = \mathbf{1}_{W^1=\hat{W}_t^1 \text{ and } W^2=\hat{W}_t^2} \Rightarrow$$
$$\mathbf{E}\rho_t(W^1, W^2, \hat{W}_t^1, \hat{W}_t^2) = Pr(W^1 = \hat{W}_t^1 \text{ and } W^2 \neq \hat{W}_t^2)$$

- Focus on structural properties of the communication system that are common regardless of these choices.

- Salient features of the problem:
  1. Domain of encoding functions $X_t^i = f_t^i(W^i, Y^{t-1})$ increases with time.
  2. Existence of common information at encoders (Y^{t-1} at time t) and private information (W^i)
  3. Decentralized, non-classical information structure (this is **not** a MDP/POMDP-like problem!)

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Introduction of pre-encoder[5]

- Equivalent encoder description:
  Each user's transmission $X_t^i = f_t^i(W^i, Y^{t-1})$ can be thought of as a two-stage process

  1. Based on available feedback Y^{t-1} select encoding functions
  $$E_t^i : \mathscr{W}^i \to \mathscr{X}^i, \qquad i = 1, 2,$$
  through a pre-encoder mapping
  $$(E_t^1, E_t^2) = h_t(Y^{t-1}).$$
  2. Generate transmitted signals by evaluating the encoding functions at W^i, i.e.,
  $$X_t^i = E_t^i(W^i), \qquad i = 1, 2.$$

[5] [Walrand and Varaiya, 1983, Nayyar and Teneketzis, 2008]

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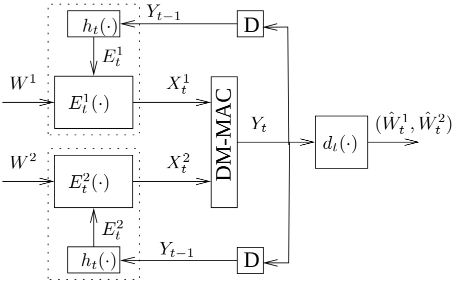
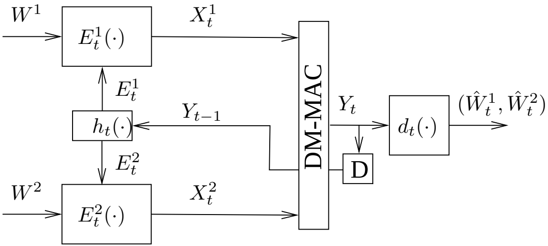


- Decentralization of information is imposed by design (h_t only uses the common information Y^{t-1} available to both encoders)
- Both encoders can evaluate each-other's encoding functions through $(E_t^1, E_t^2) = h_t(Y^{t-1})$ (can be thought of as a *fictitious* coordinator)

Transforming to a centralized control problem



- The control problem boils down to selecting encoding functions $(E_t^1, E_t^2) = h_t(Y^{t-1})$. Generation of X_t^i is a "dumb" function evaluation $X_t^i = E_t^i(W^i)$
- New equivalent design $g \stackrel{\text{def}}{=} \{f_t^1, f_t^2, d_t\}_t \Rightarrow \tilde{g} \stackrel{\text{def}}{=} \{h_t, d_t\}_t$
- Above transformation still suffers from increasing domain $\mathscr{Y}^{t-1}$ of the pre-encoder h_t, i.e., $(E_t^1, E_t^2) = h_t(Y^{t-1})$.

Introduction of information state

- We would like to summarize Y^{t-1} in a quantity (state) with time invariant domain
- Related attempts:
  1. Introduction of auxiliary variables in information theory (e.g., [Cover and Leung, 1981, Bross and Lapidoth, 2005])
  2. Form a *graph* describing the correlation structure of the messages after receiving Y^{t-1} [Venkataramanan and Pradhan, 2009]
- A more direct approach: define the random quantities

$$\Pi_t \in \mathscr{P}(\mathscr{W}^1 \times \mathscr{W}^2), \qquad t = 0, 1, 2, \ldots$$

as

$$\Pi_t(w^1, w^2) \stackrel{\text{def}}{=} Pr(W^1 = w^1, W^2 = w^2 | Y^t),$$

i.e., the posterior distribution of the message pair given the observation.

Introduction of information state

- We would like to summarize Y^{t-1} in a quantity (state) with time invariant domain
- Related attempts:
  1. Introduction of auxiliary variables in information theory (e.g., [Cover and Leung, 1981, Bross and Lapidoth, 2005])
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Introduction of information state

Lemma

1. *The quantity* Π_t *can be recursively updated as*

$$\Pi_t = \Phi(\Pi_{t-1}, E_t^1, E_t^2, Y_t), \qquad t = 1, 2, \ldots$$

2. $(\Pi_t)_t$ *is a controlled Markov process with control action* (E_t^1, E_t^2)
3. *The optimal decoder function at time* t *is only a function of* Π_t

$$(\hat{W}_t^1, \hat{W}_t^2) = d_t^*(Y^t) = d_t^*(\Pi_t)$$

4. *The average instantaneous costs are functions of* Π_{t-1}, E_t^1, E_t^2*, i.e.,*

$$\mathbf{E}\{\rho_t(W^1, W^2, \hat{W}_t^1, \hat{W}_t^2)\} = \mathbf{E}\{\Psi_t(\Pi_{t-1}, E_t^1, E_t^2)\}.$$

where Ψ_t *are known functions.*

Main structural result

Theorem

The optimal communication system for the NLF-MAC consists of

1. *Encoders of the form* $X_t^i = E_t^i(W^i)$, $i = 1, 2$, *where*

$$(E_t^1, E_t^2) = h_t(\Pi_{t-1})$$

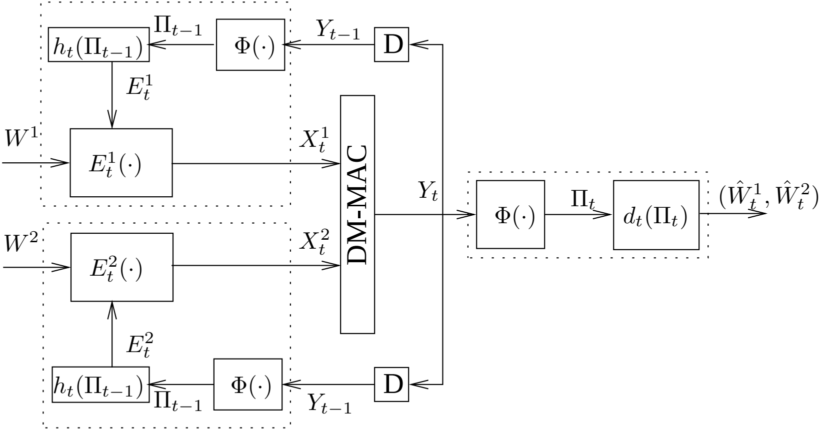
2. *A receiver that generates message estimates as*

$$(\hat{W}_t^1, \hat{W}_t^2) = d_t(\Pi_t),$$

   where d_t *is a known function.*

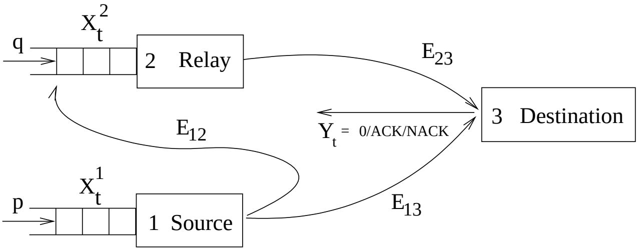
3. *The optimal* h_t *can be determined as the solution of a fix-point equation (dynamic program)*

Equivalent optimal communication system



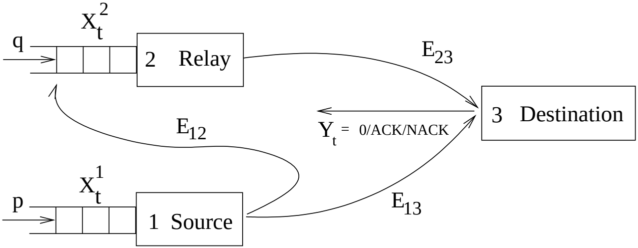
Overview

Single-relay network



- Bernoulli arrivals at Source (w.p. p) and at Relay (w.p. q)
- Packets waiting at Source's and Relay's queues $X_t = (X_t^1, X_t^2) \in \mathbb{N} \times \mathbb{N}$
- Actions $U_t^1 \in \mathscr{U}^1 \stackrel{\text{def}}{=} \{0, E_{13}, E_{12}\}$, $U_t^2 \in \mathscr{U}^2 \stackrel{\text{def}}{=} \{0, E_{23}\}$
- Simple collision model. Feedback $Y_t \in \{\emptyset, ACK, NACK\}$

Single-relay network



- Instantaneous costs are functions of energy and "delay"

$$C_t^i = \rho^i(X_t^i, U_t^i) \qquad (\text{e.g., } = X_t^i + U_t^i), \qquad i = 1,2$$

- Reasonable assumptions: $E_{12} + E_{23} < E_{13}$, $p + q < 1$, units either receive or transmit

Three scenaria of interest: Scenario A

- **Centralized** control of queues with **perfect** observation

$$(U_t^1, U_t^2) = f_t(X^{1,t}, X^{2,t}, Y^{t-1})$$

- Find a set of policies $f \stackrel{\text{def}}{=} \{f_t\}_t$ that minimize

$$J(f) = \mathbf{E}^f\{\sum_t \rho^1(X_t^1, U_t^1) + \rho^2(X_t^2, U_t^2)\}$$

- **Solution**: Centralized stochastic control problem. Can be formulated as an MDP. $(U_t^1, U_t^2) = f_t^*(X_t^1, X_t^2)$

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Three scenaria of interest: Scenario B

- Decentralized control of queues: each agent i observes only his own queue length X_t^i and both agents have a common goal (team problem)

$$U_t^i = f_t^i(X^{i,t}, Y^{t-1}), \qquad i = 1, 2$$

Find a set of policies $f \stackrel{\text{def}}{=} (f_t^1, f_t^2)_t$ that minimize

$$J(f) = \mathbf{E}^f \{ \sum_t \rho^1(X_t^1, U_t^1) + \rho^2(X_t^2, U_t^2) \},$$

- Salient features of the problem:
  1. Domain of control mappings $U_t^i = f_t^i(X^{1,t}, Y^{t-1})$ increases with time.
  2. Presence of common information (Y^{t-1} at time t) and private information ($X^{i,t}$ at time t for agent i)
  3. Decentralized, non-classical information structure (this is **not** a MDP/POMDP-like problem!)

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Structural results for the team problem

Lemma ([Vasal and Anastasopoulos, 2012])

1. *Knowledge of Y^{t-1} and $U^{i,t-1}$ reveals $U^{t-1} = (U^{1,t-1}, U^{2,t-1})$, so U^{t-1} is common knowledge (Y^{t-1} is not needed further)*

$$U_t^i = f_t^i(X^{i,t}, U^{t-1}), \qquad i = 1,2$$

2. *Optimal policy depends only on the current private state X_t^i*

$$U_t^i = f_t^i(X_t^i, U^{t-1}), \qquad i = 1,2$$

Still we have not addressed the decentralization issue and the expanding domain of f_t^i issue.

Introduction of pre-encoder[6]

Equivalent controller description:
Each agent's decision $U_t^i = f_t^i(X_t^i, U^{t-1})$ can be thought of as a two-stage process

1. Based on common info U^{t-1} select "prescription" functions $\Gamma_t^i : \mathbb{N} \to \mathscr{U}^i, \quad i = 1,2$ through the pre-encoder mapping

$$(\Gamma_t^1, \Gamma_t^2) = h_t(U^{t-1})$$

2. The actions U_t^i are determined by evaluating Γ_t^i at the private information X_t^i, i.e.,

$$U_t^i = \Gamma_t^i(X_t^i), \qquad i = 1,2$$

[6] [Walrand and Varaiya, 1983, Nayyar and Teneketzis, 2008]

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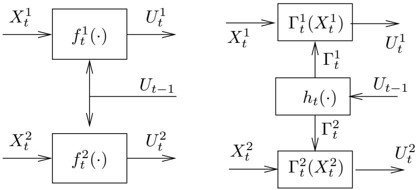
Transformation to a centralized problem



- Generation of U_t^i is a "dumb" function evaluation $U_t^i = \Gamma_t^i(X_t^i)$
- The control problem boils down to selecting prescription functions $h \stackrel{\text{def}}{=} \{h_t\}_t$,
- Both agents can evaluate each-other's prescription functions through $(\Gamma_t^1, \Gamma_t^2) = h_t(U^{t-1})$ (can be thought of as a *fictitious* controller)
- The decentralized control problem has been transformed to a centralized control problem
- Last issue to address: increasing domain $\mathscr{U}^{t-1}$ of the pre-encoder mappings h_t.

Introduction of information state

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- Define the random quantities

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Main structural result

Theorem ([Vasal and Anastasopoulos, 2012])

The original decentralized control problem is equivalent to an MDP with

- *State* Π_t
- *Control actions* $\Gamma_t \stackrel{def}{=} (\Gamma_t^1, \Gamma_t^2)$
- *Instantaneous costs* $\mathbf{E}\{\rho_t^1(X_t^1, U_t^1) + \rho_t^2(X_t^2, U_t^2) | \Pi_t, \Gamma_t\}$

Markov policies are optimal, i.e., optimal actions can be of the form

$$\Gamma_t = h_t^*(\Pi_t) \qquad \Rightarrow \qquad U_t^i = f_t^{i*}(\Pi_t, X_t^i)$$

It turns out there is a further simplification: instead of joint posterior distributions, we can use the two marginals! A general version of this result in [Nayyar et al., 2011]

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Three scenaria of interest: Scenario C

- The Source/Relay act strategically: they want to minimize their own average costs over the given time horizon

$$J^i(f) = \mathbf{E}^f\{\sum_t \rho^i(X_t^i, U_t^i)\}, \qquad i = 1,2$$

- Enlarge action space for Relay (to allow acceptance/rejection of Source packet)

$$U_t^2 \in \mathscr{U}^2 = \{0a, 0r, E_{23}\}$$

- One can study the resulting dynamic game and find Nash/sub-game perfect equilibria
- Unfortunately the equilibria of this game do not coincide with the optimal centralized solution of scenario A! (a.k.a., price of *anarchy*) Example: if optimal centralized action was $(E_{12}, 0a)$ this can never be a NE, because Relay is better off playing $0r$ (reject packet from source)

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Incentivizing cooperation

- Our approach: Devise a protocol that provides incentives to Source/Relay so that the resulting dynamic game has equilibria that coincide with the solutions of the optimal centralized problem (Scenario A)
- Introduce a state/action-dependent monetary transfer $c(X_t, U_t)$ between agents

$$\hat{\rho}^1(X_t, U_t) = \rho^1(X_t^1, U_t^1) + c(X_t, U_t)$$
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- Observe: the total societal cost is the same as in the centralized problem

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Incentivizing cooperation: basic result

- Important assumption: users know each others cost functions $\hat{\rho}^i$ (strategic behaviour does not manifest itself in desire for privacy/untrouthful revelation of cost structure)

Theorem ([Vasal and Anastasopoulos, 2013])

There exist monetary transfers $c(\cdot,\cdot)$ such that the unique Nash (sub-game perfect) equilibrium of the resulting dynamic game is exactly the optimal solution of the centralized control problem

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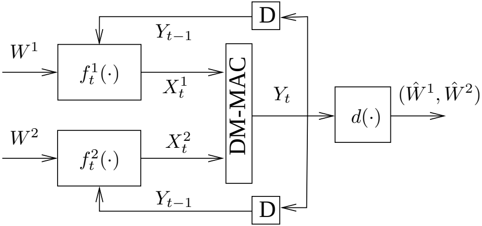
Conclusions

- A number of communications problems can be viewed as centralized/decentralized control systems
- Using ideas from Control we can derive structural results and simplify the solution of these problems
- Can handle: dynamics; cooperation (team problems); and to some extent competition (games)
- Still a lot of open problems in this area
    - Capacity-achieving / Error exponent-achieving actual communication systems (single/multi-user)
    - Single-letter capacity for MAC with feedback...

Open problem: Capacity of Multiple Access Channel with noiseless feedback



- Messages $W^i \in \{1, 2, \ldots, 2^{nR^i}\}$, $i = 1, 2$
- Transmitted symbols $X_t^i \in \mathscr{X}^i$, $i = 1, 2$, $t = 1, 2, \ldots, n$
- Received symbols $Y_t \in \mathscr{Y}$, $t = 1, 2, \ldots, n$
- Discrete-memoryless MAC (DM-MAC) $Q(y_t | x_t^1, x_t^2)$
- Message estimates $(\hat{W}^1, \hat{W}^2) \in \mathscr{W}^1 \times \mathscr{W}^2$
- Encoding functions $X_t^i = f_t^i(W^i, Y^{t-1})$, $i = 1, 2$, $t = 1, 2, \ldots, n$
- Decoding function $(\hat{W}^1, \hat{W}^2) = d(Y^n)$

Thank you!

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