New communication strategies for broadcast and interference networks

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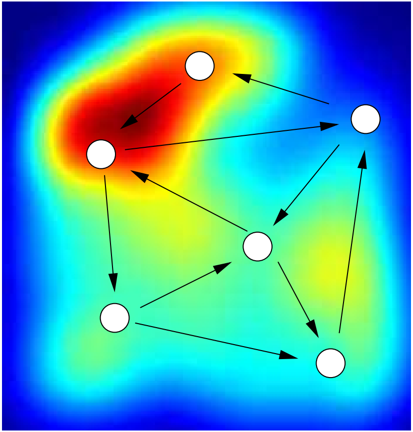
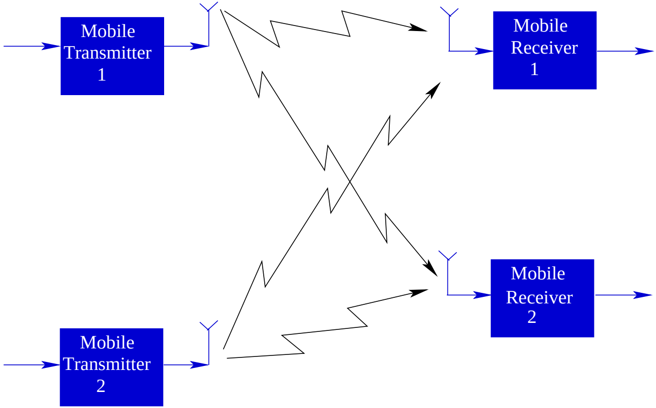
Distributed Information Coding

- Proliferation of wireless data and sensor network applications
- Supported by distributed information processing
- Information-theoretic perspective

1: Distributed Field Gathering



2: Broadcast and Interference Networks



Information and Coding theory: Tradition

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- Develop efficient communication strategies
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Coding Theory:

- Approach these limits using algebraic codes (Ex: linear codes)
- Fast encoding and decoding algorithms
- Objective: practical implementability of optimal communication systems

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- Information-theory scale: 10^{n}, n sufficiently large.

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  - Build a collection of communication systems (ensemble)
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Coding Theory Tools: Abstract algebra (groups, fields)

- Exploit algebraic structure to develop algorithms of polynomial complexity for encoding/decoding
- Study a very small ensemble at a time.

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Random Coding in networks

- Prob. distribution on a collection of codebooks (ensemble)
- Extensions of Shannon ensembles
- Lot of bad codebooks in the ensemble
- Average performance significantly affected by these bad codes
- Do not achieve optimality in general
- Many problems have remained open for decades.

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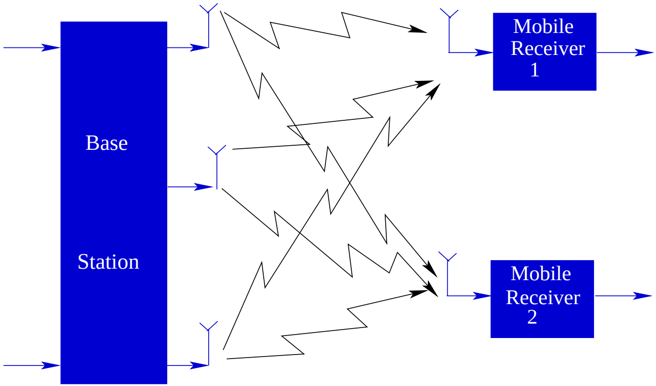
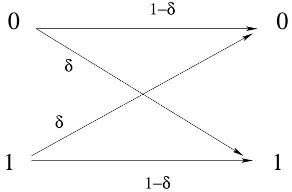
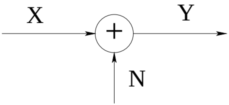
Broadcast Networks



Point-to-point communication

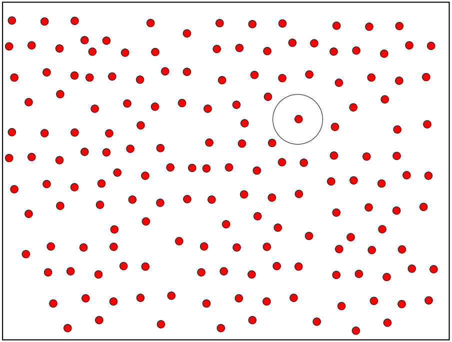
Start with Binary symmetric channel





- $N \sim Be(\delta)$, and $+$ is addition modulo 2
- Capacity $= \max_{P(x)} I(X;Y) = 1 - h(\delta)$.

Picture of an optimal code



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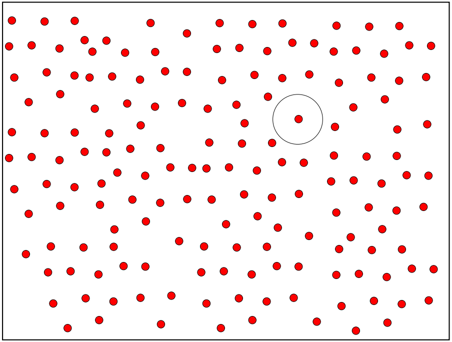

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- Equals the number of protons in the observable universe
- Named after Arthur Eddington.

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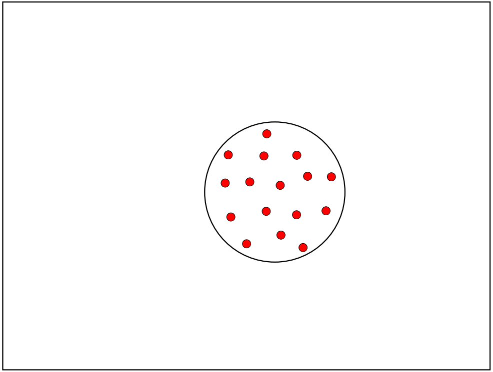
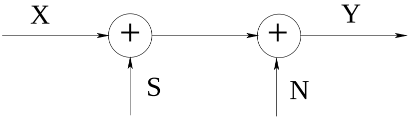


- Big circle: the set of all words with q fraction of 1's

BSC with cost constraint and interference



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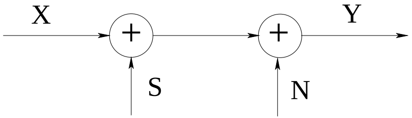

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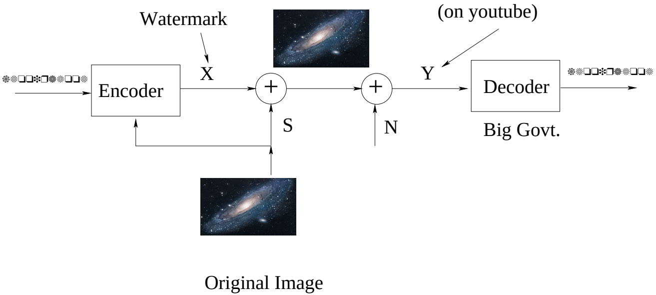
Applications

Digital watermarking, data hiding, covert communication



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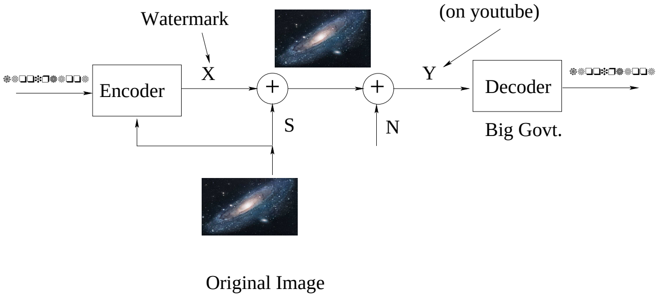

- Blind watermarking
- You want big govt. but you dont trust it too much

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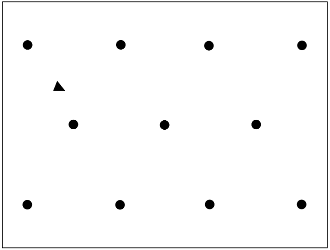
  - Gelfand-Pinsker: Nudge toward a codeword from a set
  - Q2. How large should the set be?
  - Rate of the set: $1 - h(q)$.

Picture of an optimal set of codewords



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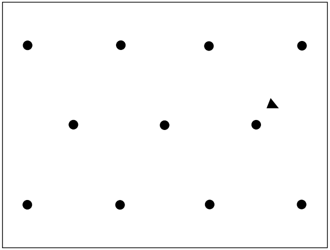

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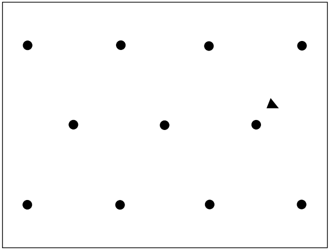

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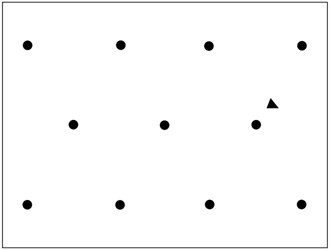

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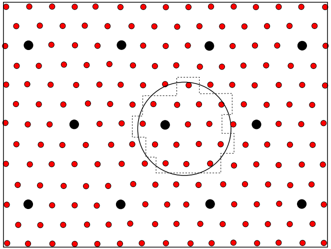
- New effective channel: $Y = U + N$ with capacity $1 - h(\delta)$

Precoding for Interference



- Rate of the composite codebook: $1 - h(\delta)$
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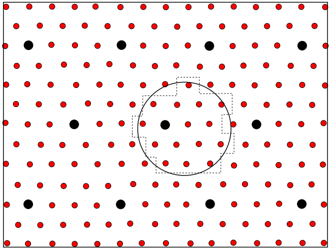

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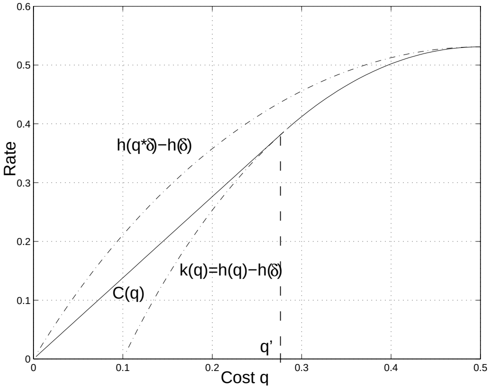
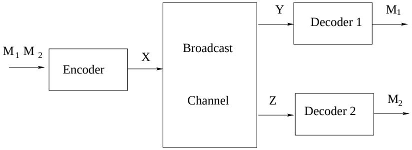
$$C(q) = \max_{P(U,X|S):Ew_H(X) \leq q} I(U;Y) - I(U;S)$$

Picture of Capacity cost function



Bottomline: Rate loss as compared to no inferference

Broadcast Channel: Cover '72



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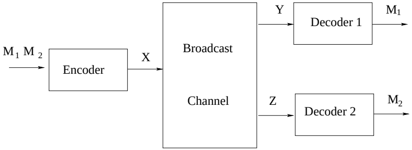

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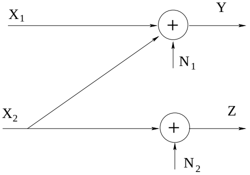
- This strategy is optimal for many special cases
- We do not know whether it is optimal in general

Example: so-called non-degraded channel



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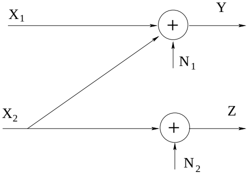

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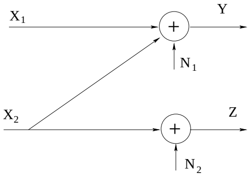

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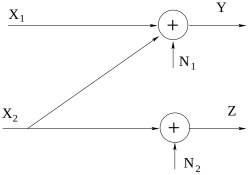

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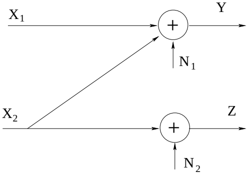

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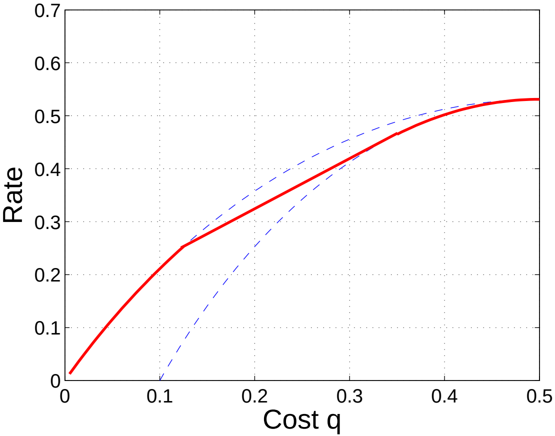
- When $q * \delta \leq \epsilon$, Rec. 1 can decode interference completely
    - a.k.a no interference $\Rightarrow R_1 = h(q * \delta) - h(\delta)$
- Otherwise, precode for X_2: $\Rightarrow R_1 = h(q) - h(\delta)$

Picture of Rate Region



Decode a *univariate* function of interference & precode for the rest

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- We can show that such a strategy is strictly suboptimal

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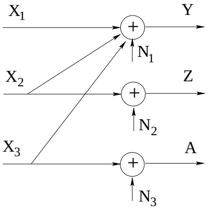

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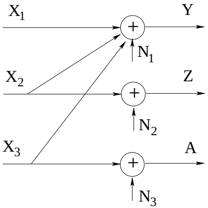

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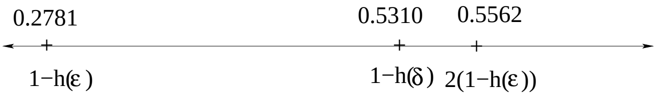
- Let $R_2 = R_3 = 1 - h(\epsilon)$, the incorrigble brutes!
- Let $\delta < \epsilon$

Deficiency of random codes

- $\delta = 0.1$ and $\epsilon = 0.2$



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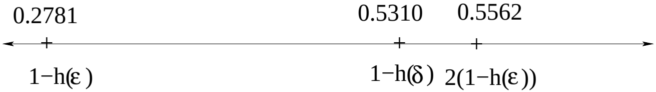

- Marton wishes to decode "full" interference: (X_2, X_3):
  - $1 - h(q * \delta) > 2(1 - h(\epsilon))$

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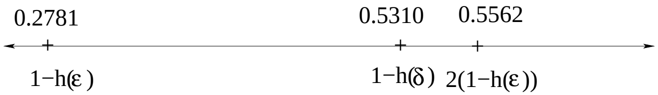

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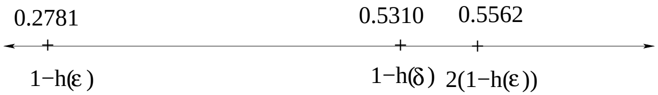

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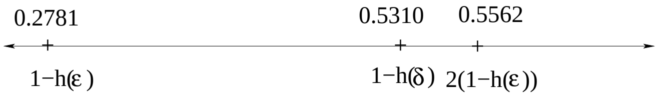

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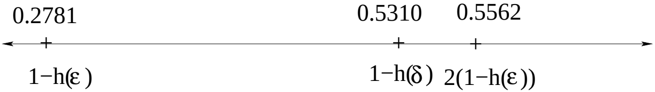

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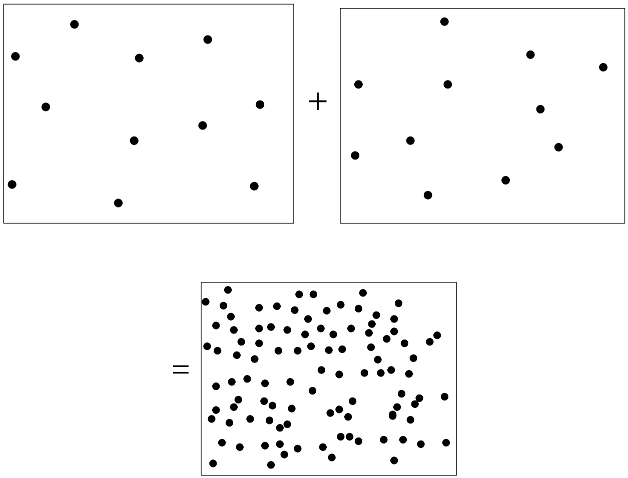
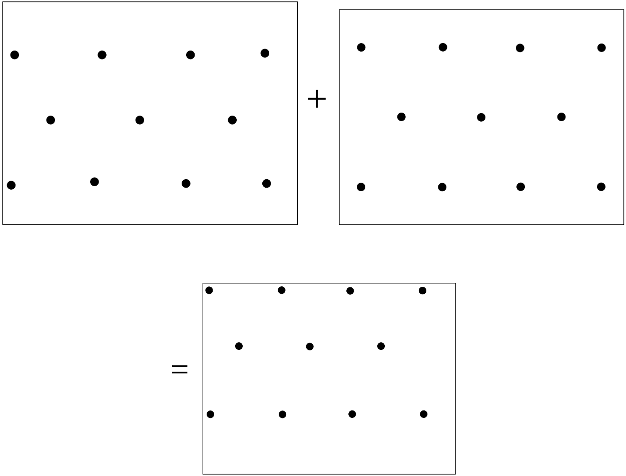
Picture of sum of two random sets



Picture of sum of two cosets of a linear code



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- Rec. 1 can decode the actual interference and subtract it off
- Then decodes her message at rate $h(q * \delta) - h(\delta)$
- $R_1 = h(q * \delta) - h(\delta)$, $R_2 = R_3 = 1 - h(\epsilon)$

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But Shannon theory is all about not getting bogged down in an example

- Objective is to develop a theory for general case

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    - In general, linear codes do not achieve Shannon capacity of an arbitrary discrete memoryless channel
- What hope do we have in using them for network communication for the arbitrary discrete memoryless case?

Thesis

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- Algebraic structure alone is not sufficient
- A right mix of algebraic structure along with non-linearity
- Nested algebraic code appears to be a universal structure

Noisy Channel Coding in Point-to-point case

- Given: Channel I/P$= X$, O/P$=Y$, with $p_{Y|X}$, and cost function $w(x)$
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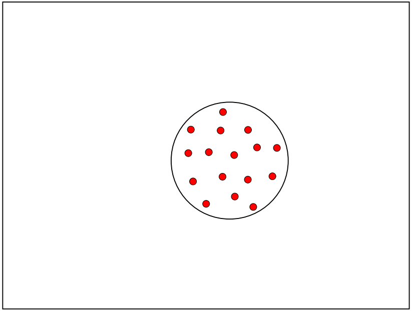
- Answer: Shannon Capacity-Cost function (Shannon '49)

$$C(W) = \max_{p_X : Ew \leq W} I(X;Y)$$

Picture of a near-optimal channel code

Obtained from Shannon ensemble



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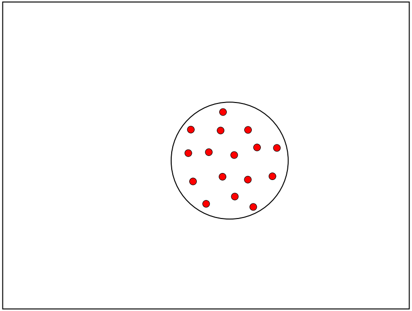

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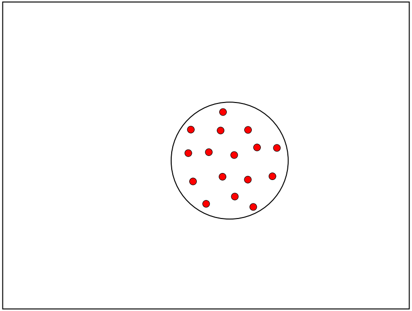

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- Shape Region $=$ Typical set
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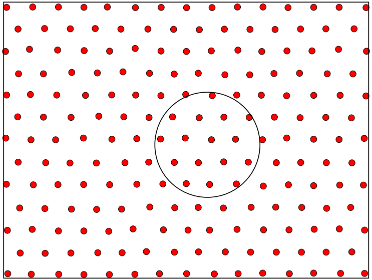
- Codeword density $=$

  $I(X;Y) - H(X) = -H(X|Y)$

New Result: An optimal linear code



- Let $|\mathcal{X}| = p$, prime no.
- $\mathcal{C}_1$ = code book
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  $= \log|\mathcal{X}| - H(X|Y)$

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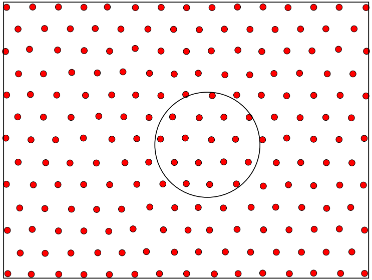

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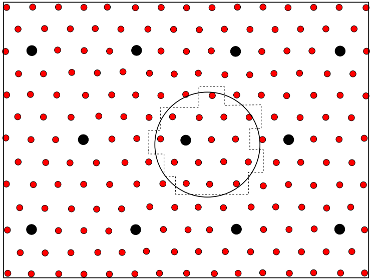
- Finite field is $\mathbb{Z}_p$
- Bounding Region $= \mathcal{X}^n$
- Density $= -H(X|Y)$

New Theorem: An optimal nested linear code



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- $\mathcal{C}_1$ fine code (red & black)
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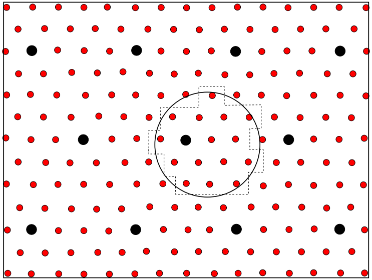

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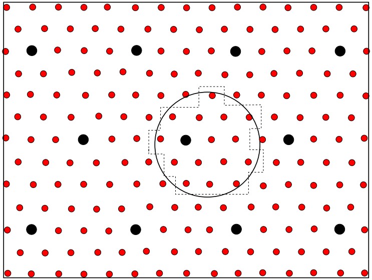

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- In general we 'embed' bivariate functions in groups

Groups - An Introduction

- G - a finite abelian group of order n
- $G \cong \mathbb{Z}_{p_1^{e_1}} \times \mathbb{Z}_{p_2^{e_2}} \cdots \times \mathbb{Z}_{p_k^{e_k}}$
- G isomorphic to direct product of possibly repeating primary cyclic groups

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Nested Group Codes

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- $(\mathcal{C}_1, \mathcal{C}_2)$ nested if $\mathcal{C}_2 \subset \mathcal{C}_1$
- We need:
  - $\mathcal{C}_1 < \mathbb{Z}_{p^r}^n$: "good" packing code
  - $\mathcal{C}_2 < \mathbb{Z}_{p^r}^n$: "good" covering code

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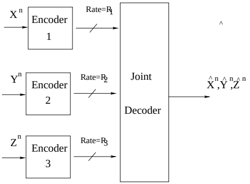
A Distributed Source Coding Problem



- Encoders observe different components of a vector source
- Central decoder receives quantized observations from the encoders
- Given source distribution p_{XYZ}
- Best known rate region - Berger-Tung Rate Region, '77

Conclusions

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